

Polynomial Dynamics

$f(z) = a_n z^n + \dots + a_0$ has a superattracting fixed pt at ∞ . wlog $a_n = 1$

Def. $f(z) \in \mathbb{C}[z]$. Filled Julia set is

$$K(f) = \{z \in \mathbb{C} \mid \{f^n(z)\}_{n \geq 1} \text{ is bounded}\} \subset \mathbb{C} = \hat{\mathbb{C}} \setminus A(f)$$

Prop. 1) $K(f)$ is compact in $\mathbb{C}$

2) $\mathbb{C} \setminus K(f)$ is connected

$$3) \partial K(f) = J(f)$$

4) $K(f)$ is the union of $J(f)$ with all Fatou components other than $A_0(\infty)$.

Proof. 1) $\exists R > 0$ st. $|z| \geq R \Rightarrow |f(z)| > 2|z|$,
 $f^n(z) \rightarrow \infty$

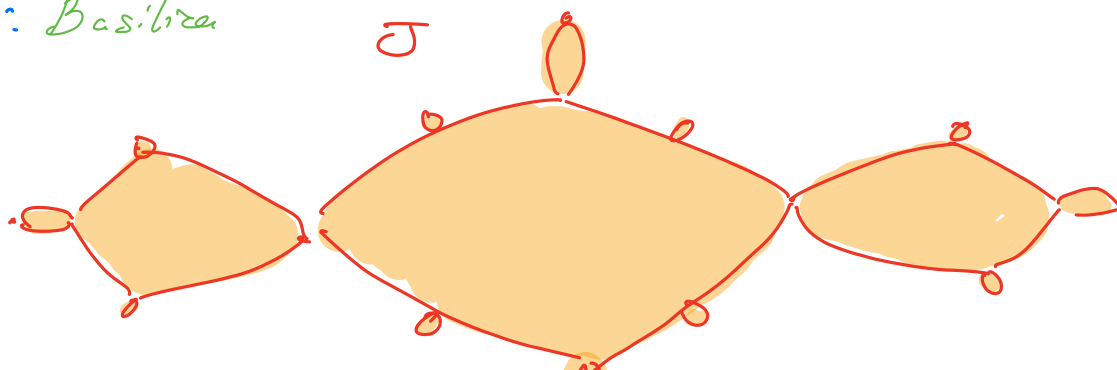
so $K \subset \mathbb{D}_R$, so bounded. Closed since $K = A^\infty$.

2) $U \subset A(\infty)$ bounded component. But then

$$\sup_{z \in U} |f^n(z)| \leq \sup_{z \in J(f)} |f^n(z)| < R \Rightarrow K = A(\infty)^\infty$$

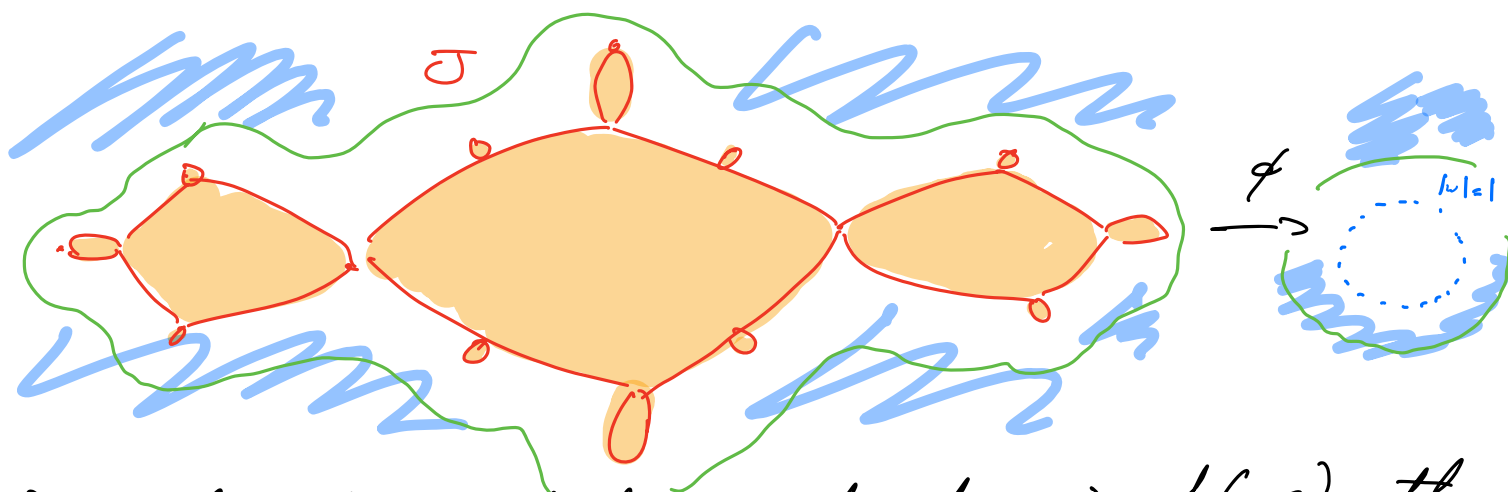
$$3, 4) \partial K(f) = \partial A(\infty) = J(f)$$

Ex: Basilica



Thm. A) $K(f)$ is connected iff $J(f)$ is separated iff all critical points lie in $K(f)$. In this case,

$$\begin{array}{ccc} A(cs) & \xrightarrow{f} & A(cs) \\ \phi \downarrow \cong & & \cong \downarrow \phi \\ \hat{C} \setminus \overline{D} & \xrightarrow{\quad} & \hat{C} \setminus \overline{D} \\ \omega & \longmapsto & \omega^n \end{array}$$



B) If all critical points lie in $A(cs)$, then $\exists$ home $\sigma: J(f) = K(f) \xrightarrow{\sim} \{0, \dots, d-1\}^N$ (Cantor set) st

$$\begin{array}{ccc} K(f) & \xrightarrow{f} & K(f) \\ \sigma \downarrow \cong & & \cong \downarrow \sigma \end{array} \quad d((a_i)_i, (b_i)_i) = d^{-\min \{j \mid a_j \neq b_j\}}$$

$$\{0, 1, \dots, d-1\}^N \xrightarrow{\sigma \circ \text{shift}} \{0, 1, \dots, d-1\}^N$$

Ex: $d=2: f_c(z) = z^2 + c$

$M := \{c \mid K(f_c) \text{ connected}\} = \{c \mid \{f_c^{-n}(0)\} \text{ bounded}\}$
Maximal set

• If $c \notin K$, then $K(f_c) = J(f_c)$ is a Cantor set.
 Proof. A)

1) $\text{Crit} \subset K(f)$: Böttcher: $A \xrightarrow{\sim} \mathbb{D}$; Conjugate with $z \mapsto \frac{1}{z}$

$$\phi: A(\infty) \xrightarrow{\sim} \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

• Define $J(f) = \partial A(\infty)$,

$$J(f) = \bigcap_{\epsilon > 0} \overline{\phi^{-1}(\{1-\epsilon < |w| \leq 1+\epsilon\})} \leftarrow \text{compact + connected}$$

$\Rightarrow J(f)$ is connected, $K(f)$ is connected.

2) $\text{Crit} \not\subset K(f)$:

• $\exists r > 1$ such st.

$$\phi^{-1}: \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}_r \rightarrow A(\infty) \text{ is}$$

defined, and $\exists v_c \in \partial \phi^{-1}(\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}_r) \subset A(\infty)$

• $v = f(c)$, $|f|(v) = r^2$, let

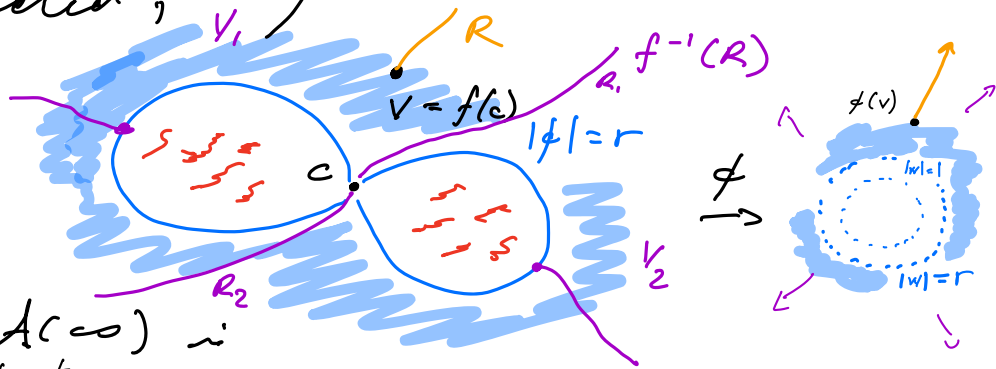
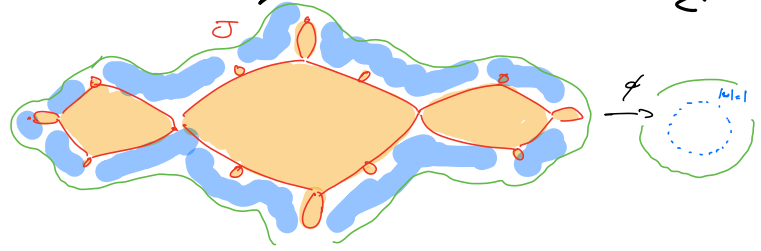
$$R = \phi^{-1} \{ t \cdot \phi(v) \mid t \in [1, \infty) \} \quad \text{external ray}$$

• $f^{-1}(R)$ will have 2 pieces, two of which, R_1, R_2 , land at c . $R_1 \cup R_2$

disconnect $\mathbb{C}$: $\mathbb{C} = (R_1 \cup R_2) \cup V_1 \cup V_2$. Each

V_i covers all of $\mathbb{C}$ except possibly $R_1 \cup R_2$,

but in particular, J and K , so $V_i \cap K \neq \emptyset$.



K, J are disconnected.

B) Prove for $f_c(z) = z^2 + c$, c is cent. value

$V = \{z \mid |f(z)| < |f(c)|\}$ is a disk

$$\text{Since } f^{-1}: D_r \xrightarrow{\sim} \frac{r}{V}^c$$

No cent. values of f_c in V , so

f_c is a covering map $f_c^{-1}(V) \rightarrow V$. Hence

$f_c^{-1}(V)$ consists of two disks D_0, D_1 .

$$K = \bigcap_{n \geq 0} f_c^{-n}(V). \quad z \in K, \quad \sigma(z)_i = \begin{cases} 0 & \text{if } f^i(z) \in D_0 \\ 1 & \text{if } f^i(z) \in D_1 \end{cases}$$

$\sigma: K \rightarrow \{0,1\}^{\mathbb{N}}$

$\{0,1\}^{\mathbb{N}}$ is compact, so we need cont., inj., surj.

Cont.: Follows from cont. of f_c

Surj.: $D_i \subset f_c^{-1}(V)$ component. $f_c^n: D_i \rightarrow V$

is a diffeomorphism, so surj. is attained.

Inj.: $\sigma(z) = \sigma(z')$.

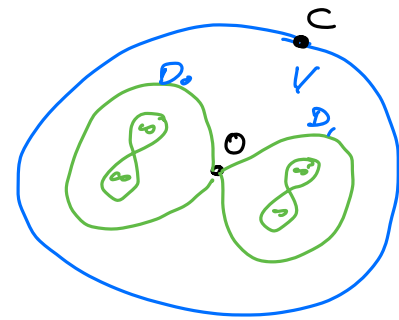
Recall that the inclusion $D_i \hookrightarrow V$ are strictly contracting (Prib):

$$d_V(x, x') < d_{D_i}(x, x').$$

Since $\overline{D_i} \subset V$ is compact,

$$d_V(x, x') \leq c d_{D_i}(x, x') \text{ for } c < 1.$$

Branches $f_c^{-1}: V \rightarrow D_0, f_c^{-1}: V \rightarrow D_1$ are



isometry, so for $V \xrightarrow{f_c} D_i \hookrightarrow V$

$$d_V(f_c^{-1}(x), f_c^{-1}(x')) \leq c d_D(f_c^{-1}(x), f_c^{-1}(x')) \\ = c \cdot d_V(x, x').$$

$$\Rightarrow d_V(z, z') \leq c \cdot d_V(f_c(z), f_c(z')) \\ \leq c^n d_V(f_c^n(z), f_c^n(z'))$$

But since the elements of $\overline{D_i}$ are labeled in V , $\text{RHS} \rightarrow 0$ and $d_V(z, z') = 0$.

Parabolic Fixed Point

$$f(z_0) = z_0, \quad \lambda = e^{2\pi i \frac{p}{q}}, \quad z_0 \in J(f)$$

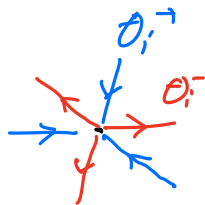
$$f(z) = \lambda z + a_2 z^2 + \dots$$

$$f^q(z) = z(1 + a z^u + \dots), \quad u = \text{multiplicity}$$

$$\theta = \arg a$$

Attracting directions: $\theta_j^+ = (\theta + (2j+1)\pi)/u$

Repelling directions: $\theta_j^- = (\theta + 2\pi j)/u$



Thm. For a parabolic fixed point,

A) $h = \frac{q}{u} v$ for some v

B) There exist u disjoint open sets $P_1^+, \dots, P_u^+$ called attracting petals s.t.

1) $P_i^+ \subset F(f)$

2) $f(P_i^+) = P_{i+pu}^+$

- 3) $P_i^+ \cong \mathbb{D}$, ∂P_i^+ is Jordan curve, ∂P_i^+ is smooth everywhere except at $z_0 \in \partial P_i^+$, and the two tangents at z_0 are at angles θ_{i-1}^- , θ_i^- (repelling)
- 4) Any orbit of f^q eventually becomes a subset of P_i^+ iff it converges monotonically to z_0 and

$$\lim_{k \rightarrow \infty} \arg(f^{kq}(z)) = \theta_i^+.$$

- 5) Every orbit which converges monotonically to z_0 eventually enters an attractive petal.
- 6) There exist a disjoint open set $P_1^-, \dots, P_n^-$ called repelling petal which satisfy B) but for f^{-1} , and w) θ_i^- and θ_i^+ switched.

- D) For any petal P , there exists a
- $$\alpha: P \rightarrow \mathbb{C} \text{ hsb and iso / its image s.t.}$$
- $$\begin{array}{ccc} P \cap f^{-1}(P) & \xrightarrow{f} & P \cap f^{-1}(P) \\ \alpha \downarrow & & \downarrow \alpha \\ \mathbb{C} & \xrightarrow{\quad} & \mathbb{C} \\ w & \xrightarrow{\quad} & w+1 \end{array} \quad \alpha(z)+1 = \alpha(f(z))$$

* n petals, m cycle, $\frac{P}{q}$ petals #

Fractal & treen: ny verlt

Ex: $f(z) = z + z^2$, $\frac{p}{q} = 1$, $u = 1$, $v = 1$

Probable loss in the street θ_1^+

$$A(z_0, \theta_i^+) = \{z \mid f^+(z) \in P_i^+ \text{ for } u\}$$

Lemma 1 (parabolic law): $A(z_0, \theta, \tau)$

* 2 $\frac{1}{2}$ left to the business

Ex: $f(z) = e^{2\pi i/3} z + z^2$, $\frac{p}{q} = \frac{1}{3}$,

$$f^3(z) = z(1+z+\dots), \quad u=3, \quad u=1$$

Ex: $f(z) = e^{2\pi i/3} z + z^3$, $\frac{f}{g} = \frac{1}{3}$

$$f^3(z) = z(1 + az^6 + \dots), \quad n = 6, \quad \omega = 2$$

Ex: $f(z) = z^{2\pi i \cdot \frac{3}{5}} z + z^3$, $\frac{p}{q} = \frac{3}{5}$, $n = 10$, $m = 2$

